On higher-power moments of $\Delta(x)$ (III)

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Abstract

Let $\Delta(x)$ be the error term of the Dirichlet divisor problem. An asymptotic formula with the error term $O(T^{53/28+\varepsilon})$ is established for the integral $\int_1^T \Delta^4(x)dx$. Similar results are also established for some other well-known error terms in the analytic number theory .

1 Introduction and main results

Let $d(n)$ denote the Dirichlet divisor function and $\Delta(x)$ denote the error term of the sum $\sum_{n\le x} d(n)$ for a large real variable x. Dirichlet first proved that $\Delta(x) = O(x^{1/2})$. The exponent $1/2$ was improved by many authors. The latest result reads

$$\Delta(x) \ll x^{131/416}(\log x)^{26947/8320}, \tag{1.1}$$

proved by Huxley[3]. It is conjectured that

$$\Delta(x) = O(x^{1/4+\varepsilon}), \tag{1.2}$$

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which is supported by the classical mean-square result

$$\int_1^T \Delta^2(x)dx = \frac{(\zeta(3/2))^4}{6\pi^2\zeta(3)} T^{3/2} + O(T\log^5 T) \tag{1.3}$$

proved by Tong[10].

Tsang[11] studied the third- and fourth-power moments of $\Delta(x)$. He proved that the asymptotic formulas

$$\int_2^T \Delta^3(x)dx = \frac{3c_1}{28\pi^3} T^{7/4} + O(T^{7/4-\delta_1+\varepsilon}) \tag{1.4}$$

and

$$\int_2^T \Delta^4(x)dx = \frac{3c_2}{64\pi^4} T^2 + O(T^{2-\delta_2+\varepsilon}) \tag{1.5}$$

hold, where $\delta_1 = 1/14$, $\delta_2 = 1/23$,

$$c_1 := \sum_{\alpha,\beta,h\in\mathbb{N}} (\alpha\beta(\alpha+\beta))^{-3/2} h^{-9/4} |\mu(h)| d(\alpha^2 h) d(\beta^2 h) d((\alpha+\beta)^2 h),$$

$$c_2 := \sum_{\substack{n,m,k,l\in\mathbb{N} \\ \sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{l}}} (nmkl)^{-3/4} d(n)d(m)d(k)d(l).$$

Recently in [12] the author proved that (1.4) holds for $\delta_1 = 1/4$. Ivić and Sargos[7] proved that (1.4) holds for $\delta_1 = 7/20$. The author got this exponent independently. However , Professor Ivić kindly informed the author that the exponent $\delta_1 = 7/20$ had already been obtained by Professor Tsang several years ago but he had never published this result.

Following Tsang's approach , in [12] the author proved that (1.5) holds for $\delta_2 = 2/41$. This approach used the method of exponential sums. Especially if the exponent pair conjecture is true, namely, if $(\varepsilon, 1/2+\varepsilon)$ is an exponent pair, then (1.5) holds for $\delta_2 = 1/14$. However, in [7] Ivić and Sargos ingeniously proved a substantially better result . They proved that (1.5) holds for $\delta_2 = 1/12$.

In this paper, combining the method of [7] and a recent deep result of Robert and Sargos[9], we shall prove the following

Theorem 1. We have

$$\int_2^T \Delta^4(x)dx = \frac{3c_2}{64\pi^4} T^2 + O(T^{53/28+\varepsilon}). \tag{1.6}$$

The theorem is also true for other error terms . Let $P(x)$ denotes the error term of the Gauss circle problem , which is an error term similar to

$\Delta(x)$. Let $a(n)$ be the Fourier coefficients of a holomorphic cusp form of weight $\kappa = 2n \geq 12$ for the full modular group and define

$$A(x) := \sum_{n\leq x}{}' a(n), \quad x \geq 2.$$

We then have the following two corollaries, which improve the previous results([2], [11], [12]).

Corollary 1. We have

$$\int_2^T P^4(x)dx = CT^2 + O(T^{53/28+\varepsilon}). \tag{1.7}$$

Corollary 2. We have

$$\int_1^T A^4(x)dx = B_\kappa T^{2\kappa} + O(T^{2\kappa-3/28+\varepsilon}). \tag{1.8}$$

Now let's consider $E(t)$, defined by

$$E(t) := \int_0^t |\zeta(\frac{1}{2}+iu)|^2 du - t\log(t/2\pi) - (2\gamma-1)t, \ \ t \geq 2. \tag{1.9}$$

Tsang[11] also studied the fourth-power moment of $E(t)$ by using Atkinson's formula[1] and proved that

$$\int_2^T E^4(t)dt = \frac{3}{8\pi}c_2T^2 + O(T^{2-\delta_3+\varepsilon}) \tag{1.10}$$

with some unspecified constant $\delta_3 > 0$.

Ivić[4] used a different way to study the higher power moments of $E(t)$. The following is his approach. Let

$$\Delta^*(x) := \frac{1}{2}\sum_{n\leq 4x}(-1)^n d(n) - x(\log x + 2\gamma - 1), \ \ x \geq 1. \tag{1.11}$$

Then for $1 \ll N \ll x$, we have[6]

$$\Delta^*(x) = \frac{1}{\pi\sqrt{2}}\sum_{n\leq N}(-1)^n d(n)n^{-3/4}x^{1/4}\cos(4\pi\sqrt{nx}-\pi/4) + O(x^{1/2+\varepsilon}N^{-1/2}) \tag{1.12}$$

Jutila[8] proved that

$$\int_0^T (E(t)-2\pi\Delta^*(\frac{t}{2\pi}))^2dt \ll T^{4/3}\log^3 T, \tag{1.13}$$

which means that $E(t)$ is well approximated by $2\pi\Delta^*(\frac{t}{2\pi})$ at least in the mean square sense. From (1.13) Ivić[4] deduced that

$$\int_0^T E^4(t)dt = (2\pi)^5\int_0^{\frac{T}{2\pi}}(\Delta^*(t))^4dt + O(T^{23/12}\log^{3/2}T). \tag{1.14}$$

Thus the fourth power moment of $E(t)$ was transformed into the fourth power moment of $\Delta^*(t)$, which can be dealt with in the same way as the fourth power moment of $\Delta(x)$. By Tsang's result[11] , Ivić deduced from (1.14) that (1.10) holds for $\delta_3 = 1/23$. In [7], Ivić and Sargos proved that one can take $\delta_3 = 1/12$.

It is easy to see that 1/12 is the limit of this approach since it is the limit of Jutila's result (1.13). In this paper, we shall use a different way to prove the following

Theorem 2. We have

$$\int_2^T E^4(t)dt = \frac{3}{8\pi}c_2T^2 + O(T^{53/28+\varepsilon}). \tag{1.15}$$

Remark. The proof of Theorem 2 doesn't use (1.13) and it is actually a generalization of the approach used in the author [13]. In [14] the author used a similar method to study the third power moment of $E(t)$.

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Notations. Throughout this paper, $[x]$ denotes the integer part of x, $\|x\|$ denotes the distance from x to the integer nearest to x, $n \sim N$ means $N < n \le 2N$, $n \asymp N$ means $C_1N < n \le C_2N$ for positive constants $C_1 < C_2$. ε always denotes a small positive constant which may be different at different places. We shall use the estimate $d(n) \ll n^\varepsilon$ freely.

2 The spacing problem of the square roots

In the proofs of Theorem 1 and 2 , the sums and differences of square roots will appear in the exponential. Thus we should study the spacing problem of the square roots.

We need the following Lemmas. Lemma 1 is a special case of a new result proved in Robert and Sargos[9], which also plays an important role in this paper. Lemma 2 is Lemma 3 of Tsang [11]. Lemma 3 provides an upper bound of the number of solutions of the inequality

$$|n_1^{1/2} + n_2^{1/2} \pm n_3^{1/2} - n_4^{1/2}| < \Delta, \quad n_j \sim N_j (j = 1, 2, 3, 4), \tag{2.1}$$

where $N_j \geq 2 (j = 1, 2, 3, 4)$ are real numbers . Lemma 4 is essentially Lemma 3 of Ivić and Sargos[7], but we added the case $\alpha \ll 1$. Lemma 5 is essentially Lemma 5 of [7], but the term $K \min(M, M', L)$ therein is superfluous since we add the condition $|\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l}| > 0$ in Lemma 5, and so we give a new proof here. Lemma 6 is Lemma 6 of [7].

Lemma 1. Suppose $N \geq 2$, $\Delta > 0$. Let $\mathcal{A}(N; \Delta)$ denote the number of solutions of the inequality

$$|n_1^{1/2} + n_2^{1/2} - n_3^{1/2} - n_4^{1/2}| < \Delta, \quad n_j \sim N (j = 1, 2, 3, 4),$$

then

$$\mathcal{A}(N; \Delta) \ll (\Delta N^{7/2} + N^2) N^{\varepsilon}.$$

Lemma 2. If $n, m, k, l \in \mathbb{N}$ such that $\sqrt{n} + \sqrt{m} \pm \sqrt{k} - \sqrt{l} \neq 0$, then respectively,

$$|\sqrt{n} + \sqrt{m} \pm \sqrt{k} - \sqrt{l}| \gg \max(n, m, k, l)^{-7/2}.$$

Lemma 3. Suppose $N_j \geq 2 (j = 1, 2, 3, 4)$, $\Delta > 0$. Let $\mathcal{A}_{\pm}(N_1, N_2, N_3, N_4; \Delta)$ denote the number of solutions of the inequality (2.1), then we have

$$\mathcal{A}_{\pm}(N_1, N_2, N_3, N_4; \Delta) \ll \prod_{j=1}^{4} (\Delta^{1/4} N_j^{7/8} + N_j^{1/2}) N_j^{\varepsilon}.$$

Proof. We shall use a combinatorial argument to prove this Lemma. Let $\{a_i\}$ and $\{b_i\}$ be two finite sequences of real numbers. Let $\Delta > 0$. Suppose u_0 and J(a positive integer) are chosen so that $\{a_i\} \subset (u_0, u_0 + J\Delta]$, $\{b_i\} \subset (u_0, u_0 + J\Delta]$. Divide this interval into the abutting subintervals $I_j := (u_0 + j\Delta, u_0 + (j+1)\Delta]$ for $j = 0, 1, \cdots, J-1$ and then let

$$N_j(A) := \#\{i : a_i \in I_j\}, N_j(B) := \#\{i : b_i \in I_j\}.$$

If $|a_r - b_s| \le \Delta$, then either both a_r and b_s lie in the same subinterval I_j, or they lie in adjacent subintervals I_j and I_{j+1}. Hence

$$\begin{aligned}
&\#\{(r,s) : |a_r - b_s| \le \Delta\} \\
&\le \sum_j N_j(A)N_j(B) + \sum_j N_j(A)N_{j+1}(B) + \sum_j N_{j+1}(A)N_j(B) \\
&\le 3(\sum_j N_j(A)^2)^{1/2}(\sum_j N_j(B)^2)^{1/2}
\end{aligned}$$

by Cauchy-Schwarz's inequality. On the other hand , we have

$$\begin{aligned}
&\sum_j N_j(A)^2 = \sum_j \#\{(r,r') : a_r, a_{r'} \in I_j\} \\
&\le \#\{(r,r') : |a_r - a_{r'}| \le \Delta\},
\end{aligned}$$

and similarly for $\sum_j N_j(B)^2$. Thus we get the bound

$$\begin{aligned}
&\#\{(r,s) : |a_r - b_s| \le \Delta\} \qquad (2.2) \\
&\le 3(\#\{(r,r') : |a_r - a_{r'}| \le \Delta\})^{1/2}(\#\{(s,s') : |b_s - b_{s'}| \le \Delta\})^{1/2}.
\end{aligned}$$

Suppose $n_j, n_j' \sim N_j (j = 1,2,3,4)$. Applying (2.2) to the sequences $A = \{\sqrt{n_1} + \sqrt{n_2}\}$ and $B = \{\sqrt{n_3} + \sqrt{n_4}\}$, we get

$$\mathcal{A}_-(N_1, N_2, N_3, N_4) = \#\{(n_1, n_2, n_3, n_4) : |n_1^{1/2} + n_2^{1/2} - n_3^{1/2} - n_4^{1/2}| \le \Delta\} \qquad (2.3)$$

$$\begin{aligned}
&\le 3(\#\{(n_1, n_2, n_1', n_2') : |n_1^{1/2} + n_2^{1/2} - n_1'^{1/2} - n_2'^{1/2}| \le \Delta\})^{1/2} \\
&\quad \times(\#\{(n_3, n_4, n_3', n_4') : |n_3^{1/2} + n_4^{1/2} - n_3'^{1/2} - n_4'^{1/2}| \le \Delta\})^{1/2}.
\end{aligned}$$

Using the previous bound to the sequences $A_1 = \{n_1^{1/2} - n_1'^{1/2}\}, B_1 = \{n_2^{1/2} - n_2'^{1/2}\}$, and $A_2 = \{n_3^{1/2} - n_3'^{1/2}\}, B_2 = \{n_4^{1/2} - n_4'^{1/2}\}$, respectively, we get

$$\mathcal{A}_-(N_1, N_2, N_3, N_4) \le 9 \prod_{j=1}^{4} \mathcal{A}_-(N_j, N_j, N_j, N_j)^{1/4}, \qquad (2.4)$$

which combining Lemma 1 gives Lemma 3 for the case "-". The proof for the case "+" is similar.

$\square$

Lemma 4. Suppose $K \geq 10$, $\alpha, \beta \in \mathbb{R}$, $2K^{-1/2} \leq |\alpha| \ll K^{1/2}$ and $0 < \delta < 1/2$. Then we have

$$\#\{k \sim K : \|\beta + \alpha\sqrt{k}\| < \delta\} \ll K\delta + K^{1/2+\varepsilon}.$$

Proof. Without loss of generality, suppose $\alpha > 0$. Let $\mathcal{N} = \#\{k \sim K : \|\beta + \alpha\sqrt{k}\| < \delta\}$. If $1 \ll \alpha \ll K^{1/2}$, from Lemma 3 of Ivić and Sargos[7] we get

$$\mathcal{N} \ll K\delta + |\alpha|^{1/2}K^{1/4+\varepsilon} + K^{1/2+\varepsilon} \ll K\delta + K^{1/2+\varepsilon}.$$

Now suppose $2K^{-1/2} \leq \alpha \ll 1$. Since $\|t\|$ is a periodic function with period 1, we suppose $0 < \beta \leq 1$. If $\|\beta + \alpha\sqrt{k}\| < \delta$, then there exists a unique $l \in [\alpha\sqrt{K}, 2\alpha\sqrt{K} + 2]$ such that

$$(l - \beta - \delta)^2/\alpha^2 < k \leq (l - \beta + \delta)^2/\alpha^2,$$

which implies

$$\begin{aligned}
\mathcal{N} \quad & \ll \sum_{l \sim \alpha\sqrt{K}} \big([(l-\beta+\delta)^2/\alpha^2] - [(l-\beta-\delta)^2/\alpha^2]\big) \\
& \ll \sum_{l \sim \alpha\sqrt{K}} \big((l-\beta+\delta)^2/\alpha^2 - (l-\beta-\delta)^2/\alpha^2 + 1\big) \\
& \ll K\delta + K^{1/2}
\end{aligned}$$

if we notice $\alpha \ll 1$. $\square$

Lemma 5. Suppose $1 \leq N \leq M, 1 \leq L \leq K$, $N \leq L$, $M \asymp K$, $0 < \Delta \ll K^{1/2}$. Let $\mathcal{A}_1(N, M, K, L; \Delta)$ denote the number of solutions of the inequality

$$0 < |\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l}| < \Delta$$

with $n \sim N, m \sim M, k \sim K, l \sim L$. Then we have

$$\mathcal{A}_1(N, M, K, L; \Delta) \ll \Delta K^{1/2}NML + NLK^{1/2+\varepsilon}.$$

Especially if $\Delta K^{1/2} \gg 1$, then

$$\mathcal{A}_1(N, M, K, L; \Delta) \ll \Delta K^{1/2}NML.$$

Proof. If (n, m, k, l) satisfies $|\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l}| < \Delta$, then we get

$$m = k + 2k^{1/2}(\sqrt{l} - \sqrt{n}) + (\sqrt{l} - \sqrt{n})^2 + u$$

with $|u| \le C\Delta K^{1/2}$ for some absolute constant $C > 0$. Hence $\mathcal{A}_1(N, M, K, L; \Delta)$ does not exceed the number of solutions of the inequality

$$|2k^{1/2}(\sqrt{l} - \sqrt{n}) + (\sqrt{l} - \sqrt{n})^2 + k - m| < C\Delta K^{1/2} \tag{2.5}$$

with $n \sim N, m \sim M, k \sim K, l \sim L$.

If $\Delta K^{1/2} \gg 1$, then for fixed (n, k, l), the number of m for which (2.5) holds is $\ll 1 + \Delta K^{1/2} \ll \Delta K^{1/2}$ if we notice $K \asymp M$. Hence

$$\mathcal{A}_1(N, M, K, L; \Delta) \ll \Delta K^{1/2} NML.$$

Now suppose $\Delta K^{1/2} \le 1/4C$. For fixed (n, k, l), there is at most one m such that (2.14) holds. If such m exists, then we have

$$\|2k^{1/2}(\sqrt{l} - \sqrt{n}) + (\sqrt{l} - \sqrt{n})^2\| < C\Delta K^{1/2}. \tag{2.6}$$

We shall use Lemma 4 to bound the number of solutions of (2.6) with $\alpha = 2(\sqrt{l} - \sqrt{n}), \beta = (\sqrt{l} - \sqrt{n})^2$. Let $\mathcal{C}_1$ denote the number of solutions of (2.6) with $|\alpha| \ge 2K^{-1/2}$, and $\mathcal{C}_2$ the number of solutions with $|\alpha| < 2K^{-1/2}$, respectively. By Lemma 4 we get

$$\mathcal{C}_1 \ll \Delta K^{1/2} NML + NLK^{1/2+\varepsilon}$$

if we notice $M \asymp K$. Now we estimate $\mathcal{C}_2$. From $|\alpha| < 2K^{-1/2}$, we get $N \asymp L$. If $l = n$, then from (2.5) we get $k = m$. This contradicts to $|\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l}| > 0$. Thus $l \ne n$. From

$$2K^{-1/2} > |\sqrt{l} - \sqrt{n}| = \frac{|l-n|}{\sqrt{l} + \sqrt{n}} \ge \frac{1}{\sqrt{l} + \sqrt{n}} \ge 1/2\sqrt{2L}$$

we get $L \gg K$ and thus $N \asymp M \asymp K \asymp L$. So we have

$$\mathcal{C}_2 \ll \#\{(l, n) : |\alpha| < 2K^{-1/2}\} \times \#\{k\} \ll K^2,$$

which can be absorbed into the estimate of $\mathcal{C}_1$. This completes the proof of Lemma 5. $\square$

Lemma 6. Suppose $1 \le N \le M \le K \asymp L$, $0 < \Delta \ll L^{1/2}$. Let $\mathcal{A}_2(N, M, K, L; \Delta)$ denote the number of solutions of the inequality

$$|\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{l}| < \Delta$$

with $n \sim N, m \sim M, k \sim K, l \sim L$. Then we have

$$\mathcal{A}_2(N, M, K, L; \Delta) \ll \Delta L^{1/2} NMK + NMK^{1/2+\varepsilon}.$$

Especially if $\Delta L^{1/2} \gg 1$, then

$$\mathcal{A}_2(N, M, K, L; \Delta) \ll \Delta L^{1/2} NMK.$$

3 Proof of Theorem 1

Suppose $T \geq 10$. It suffices to evaluate the integral $\int_T^{2T} \Delta^4(x)dx$. Suppose $y = T^{3/4}$. For any $T \leq x \leq 2T$, by the truncated Voronoi's formula we get

$$\Delta(x) = \frac{1}{\sqrt{2}\pi}\mathcal{R} + (x^{1/2+\varepsilon}y^{-1/2}), \tag{3.1}$$

where

$$\mathcal{R} := \mathcal{R}(x) = x^{1/4}\sum_{n\leq y}\frac{d(n)}{n^{3/4}}\cos(4\pi\sqrt{xn} - \frac{\pi}{4}).$$

We have

$$\begin{aligned}\int_T^{2T} \Delta^4(x)dx &= \frac{1}{4\pi^4}\int_T^{2T} \mathcal{R}^4 dx + O(T^{9/4+\varepsilon}y^{-1/2} + T^{3+\varepsilon}y^{-2}) \\ &= \frac{1}{4\pi^4}\int_T^{2T} \mathcal{R}^4 dx + O(T^{15/8+\varepsilon}).\end{aligned} \tag{3.2}$$

Let

$$g = g(n,m,k,l) := (nmkl)^{-\frac{3}{4}}d(n)d(m)d(k)d(l), \text{for } \ n,m,k,l \leq y,$$

and $g = 0$ otherwise.

The equation (3.4) of Tsang[11] reads

$$\mathcal{R}_1^4 = S_1(x) + S_2(x) + S_3(x) + S_4(x), \tag{3.3}$$

where

$$\begin{aligned}S_1(x) &:= \frac{3}{8}\sum_{\sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{l}} gx, \\ S_2(x) &:= \frac{3}{8}\sum_{\sqrt{n}+\sqrt{m}\neq\sqrt{k}+\sqrt{l}} gx\cos(4\pi(\sqrt{n}+\sqrt{m}-\sqrt{k}-\sqrt{l})\sqrt{x}), \\ S_3(x) &:= \frac{1}{2}\sum gx\sin(4\pi(\sqrt{n}+\sqrt{m}+\sqrt{k}-\sqrt{l})\sqrt{x}), \\ S_4(x) &:= -\frac{1}{8}\sum gx\cos(4\pi(\sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{l})\sqrt{x}).\end{aligned}$$

From (3.7) of [11] we get

$$\int_T^{2T} S_1(x)dx = \frac{3c_2}{8}\int_T^{2T} xdx + O(T^{2-3/16+\varepsilon}). \tag{3.4}$$

From the first derivative test we get

$$\int_T^{2T} S_4(x)dx \ll T^{3/2+\varepsilon}y^{1/2} \ll T^{15/8+\varepsilon}. \tag{3.5}$$

Now let us consider the contribution of $S_2(x)$. By the first derivative test we get

$$\int_T^{2T} S_2(x)dx \ll \sum_{\substack{n,m,k,l\le y \\ \sqrt{n}+\sqrt{m}\ne\sqrt{k}+\sqrt{l}}} g \min\left(T^2, \frac{T^{\frac{3}{2}}}{|\sqrt{n}+\sqrt{m}-\sqrt{k}-\sqrt{l}|}\right) \tag{3.6}$$

$$\ll T^{\varepsilon}G(N,M,K,L),$$

where

$$G(N,M,K,L) = \sum_1 g \min\left(T^2, \frac{T^{\frac{3}{2}}}{|\sqrt{n}+\sqrt{m}-\sqrt{k}-\sqrt{l}|}\right),$$
$$SC(\Sigma_1): \sqrt{n}+\sqrt{m}\ne\sqrt{k}+\sqrt{l}, 1\le N\le M\le y, 1\le L\le K\le y,$$
$$N\le L, n\sim N, m\sim M, k\sim K, l\sim L.$$

If $M\ge 100K$, then $|\sqrt{n}+\sqrt{m}-\sqrt{k}-\sqrt{l}|\gg M^{1/2}$, so the trivial estimate yields

$$G(N,M,K,L) \ll \frac{T^{\frac{3}{2}+\varepsilon}NMKL}{(NMKL)^{3/4}M^{1/2}} \ll T^{\frac{3}{2}+\varepsilon}y^{\frac{1}{2}} \ll T^{15/8+\varepsilon}.$$

If $K>100M$, we get the same estimate. So later we always suppose that $M\asymp K$.

Let $\eta=\sqrt{n}+\sqrt{m}-\sqrt{k}-\sqrt{l}$. Write

$$G(N,M,K,L,R) = G_1+G_2+G_3, \tag{3.7}$$

where

$$G_1 := T^2 \sum_{|\eta|\le T^{-1/2}} g,$$
$$G_2 := T^{\frac{3}{2}} \sum_{T^{-1/2}<|\eta|\le 1} g|\eta|^{-1},$$
$$G_3 := T^{\frac{3}{2}} \sum_{|\eta|\gg 1} g|\eta|^{-1}.$$

We estimate G_1 first. From $|\eta| \le T^{-1/2}$ we get $M \asymp K \gg T^{1/7}$ via Lemma 2. By Lemma 5 we get

$$\begin{aligned}
G_1 &\ll \frac{T^{2+\varepsilon}}{(NMKL)^{3/4}}\mathcal{A}_1(N,M,K,L;T^{-1/2}) \qquad (3.8)\\
&\ll \frac{T^{2+\varepsilon}}{(NMKL)^{3/4}}\left(T^{-1/2}K^{1/2}NML + NLK^{1/2}\right)\\
&\ll T^{3/2+\varepsilon}(NL)^{1/4} + T^{2+\varepsilon}(NL)^{1/4}K^{-1}\\
&\ll T^{3/2+\varepsilon}y^{1/2} + T^{2+\varepsilon}(NL)^{1/4}K^{-1}\\
&\ll T^{15/8+\varepsilon} + T^{2+\varepsilon}(NL)^{1/4}K^{-1}.
\end{aligned}$$

By Lemma 3 we get(notice $N \le L \le K$)

$$\begin{aligned}
G_1 &\ll \frac{T^{2+\varepsilon}}{(NMKL)^{3/4}}\mathcal{A}_-(N,M,K,L;T^{-1/2}) \qquad (3.9)\\
&\ll \frac{T^{2+\varepsilon}}{(NMKL)^{3/4}}\left(T^{-1/8}N^{7/8}+N^{1/2}\right)\left(T^{-1/8}L^{7/8}+L^{1/2}\right)\left(T^{-1/4}K^{7/4}+K\right)\\
&\ll T^{2+\varepsilon}\left(T^{-1/8}N^{1/8}+N^{-1/4}\right)\left(T^{-1/8}L^{1/8}+L^{-1/4}\right)\left(T^{-1/4}K^{1/4}+K^{-1/2}\right)\\
&\ll T^{2+\varepsilon}\left(T^{-1/4}(NL)^{1/8}+T^{-1/8}L^{1/8}N^{-1/4}+(NL)^{-1/4}\right)\left(T^{-1/4}K^{1/4}+K^{-1/2}\right)\\
&\ll T^{2+\varepsilon}T^{-1/4}(NL)^{1/8}\left(T^{-1/4}K^{1/4}+K^{-1/2}\right)\\
&\quad + T^{2+\varepsilon}\left(T^{-1/8}L^{3/8}(NL)^{-1/4}+(NL)^{-1/4}\right)\left(T^{-1/4}K^{1/4}+K^{-1/2}\right)\\
&\ll T^{3/2+\varepsilon}y^{1/2}+T^{7/4+\varepsilon}K^{-1/4}\\
&\quad + T^{2+\varepsilon}\left(T^{-1/4}K^{1/4}+K^{-1/2}\right)\left(T^{-1/8}K^{3/8}+1\right)(NL)^{-1/4}\\
&\ll T^{15/8+\varepsilon}+T^{2+\varepsilon}K^{-1/2}\left(T^{-1/4}K^{3/4}+1\right)\left(T^{-1/8}K^{3/8}+1\right)(NL)^{-1/4}\\
&\ll T^{15/8+\varepsilon}+T^{2+\varepsilon}K^{-1/2}\left(T^{-3/8}K^{9/8}+1\right)(NL)^{-1/4}.
\end{aligned}$$

From (3.8) and (3.9) we get

$$\begin{aligned}
G_1 &\ll T^{15/8+\varepsilon}+T^{2+\varepsilon}\min\left((NL)^{1/4}K^{-1},K^{-1/2}\left(T^{-3/8}K^{9/8}+1\right)(NL)^{-1/4}\right) \qquad (3.10)\\
&\ll T^{15/8+\varepsilon}+T^{2+\varepsilon}\left((NL)^{1/4}K^{-1}\right)^{1/2}\left(K^{-1/2}\left(T^{-3/8}K^{9/8}+1\right)(NL)^{-1/4}\right)^{1/2}\\
&\ll T^{15/8+\varepsilon}+T^{2+\varepsilon}K^{-3/4}(T^{-3/16}K^{9/16}+1)\\
&\ll T^{15/8+\varepsilon}+T^{2+\varepsilon}K^{-3/4} \ll T^{53/28+\varepsilon}
\end{aligned}$$

if we notice $K \gg T^{1/7}$.

Now we estimate G_2. By a splitting argument we get that the estimate

$$G_2 \ll \frac{T^{3/2+\varepsilon}}{(NMKL)^{3/4}\delta} \sum_{\substack{\delta<|\eta|\le 2\delta \\ \eta\neq 0}} 1 \tag{3.11}$$

holds for some $T^{-1/2} \le \delta \le 1$. By Lemma 5 we get that

$$\begin{aligned} G_2 &\ll \frac{T^{3/2+\varepsilon}}{(NMKL)^{3/4}\delta} \mathcal{A}_1(N, M, K, L; 2\delta) \\ &\ll \frac{T^{3/2+\varepsilon}}{(NMKL)^{3/4}\delta} (\delta K^{1/2} NML + NLK^{1/2}) \\ &\ll T^{3/2+\varepsilon} y^{1/2} + T^{3/2+\varepsilon}(K\delta)^{-1}(NL)^{1/4} \\ &\ll T^{15/8+\varepsilon} + T^{3/2+\varepsilon}(K\delta)^{-1}(NL)^{1/4}. \end{aligned} \tag{3.12}$$

By Lemma 3 we get(notice $N \le L \le K$)

$$\begin{aligned} G_2 &\ll \frac{T^{3/2+\varepsilon}}{(NMKL)^{3/4}\delta} \mathcal{A}_-(N, M, K, L; 2\delta) \\ &\ll \frac{T^{3/2+\varepsilon}}{(NMKL)^{3/4}\delta} (\delta^{1/4} N^{7/8} + N^{1/2})(\delta^{1/4} L^{7/8} + L^{1/2})(\delta^{1/2} K^{7/4} + K) \\ &\ll T^{3/2+\varepsilon}(N^{1/8} + N^{-1/4}\delta^{-1/4})(L^{1/8} + L^{-1/4}\delta^{-1/4})(K^{1/4} + K^{-1/2}\delta^{-1/2}) \\ &\ll T^{3/2+\varepsilon} \left((NL)^{1/8} + L^{1/8}N^{-1/4}\delta^{-1/4} + (NL)^{-1/4}\delta^{-1/2}\right)(K^{1/4} + K^{-1/2}\delta^{-1/2}) \\ &\ll T^{3/2+\varepsilon}(NL)^{1/8}K^{1/4} + T^{3/2+\varepsilon}(NL)^{1/8}K^{-1/2}\delta^{-1/2} \\ &\quad + T^{3/2+\varepsilon}(K^{1/4} + K^{-1/2}\delta^{-1/2})(L^{3/8}\delta^{1/4} + 1)(NL)^{-1/4}\delta^{-1/2} \\ &\ll T^{3/2+\varepsilon} y^{1/2} + T^{3/2+\varepsilon}\delta^{-1/2} + T^{3/2+\varepsilon}K^{-1/2}\delta^{-1}(K^{3/4}\delta^{1/2} + 1)(K^{3/8}\delta^{1/4} + 1)(NL)^{-1/4} \\ &\ll T^{15/8+\varepsilon} + T^{3/2+\varepsilon}K^{-1/2}\delta^{-1}(K^{9/8}\delta^{3/4} + 1)(NL)^{-1/4}, \end{aligned} \tag{3.13}$$

where the bound $\delta \gg T^{-1/2}$ was used to the term $T^{3/2+\varepsilon}\delta^{-1/2}$.

From (3.12) and (3.13) we get

$$\begin{aligned} G_2 &\ll T^{15/8+\varepsilon} + \frac{T^{3/2+\varepsilon}}{\delta} \min\left(\frac{(NL)^{1/4}}{K}, \frac{K^{9/8}\delta^{3/4} + 1}{K^{1/2}(NL)^{1/4}}\right) \\ &\ll T^{15/8+\varepsilon} + \frac{T^{3/2+\varepsilon}}{\delta} \left(\frac{(NL)^{1/4}}{K}\right)^{1/2} \left(\frac{K^{9/8}\delta^{3/4} + 1}{K^{1/2}(NL)^{1/4}}\right)^{1/2} \\ &\ll T^{15/8+\varepsilon} + T^{3/2+\varepsilon}\delta^{-1}K^{-3/4}(K^{9/16}\delta^{3/8} + 1). \end{aligned} \tag{3.14}$$

If $\delta \gg K^{-3/2}$, then (3.14) implies(recall $\delta \gg T^{-1/2}$)

$$G_2 \ll T^{15/8+\varepsilon} + T^{3/2+\varepsilon} K^{-3/16} \delta^{-5/8} \ll T^{15/8+\varepsilon}. \tag{3.15}$$

If $\delta \ll K^{-3/2}$, then (3.14) becomes

$$G_2 \ll T^{15/8+\varepsilon} + T^{3/2+\varepsilon} \delta^{-1} K^{-3/4}. \tag{3.16}$$

Since we have $\delta \gg K^{-7/2}$ by Lemma 2 and $\delta \gg T^{-1/2}$, we get $\delta^{-1} \ll \min(K^{7/2}, T^{1/2})$ and thus from (3.16) we get

$$\begin{aligned} G_2 &\ll T^{15/8+\varepsilon} + \min(T^{2+\varepsilon} K^{-3/4}, T^{3/2+\varepsilon} K^{11/4}) \qquad (3.17) \\ &\ll T^{15/8+\varepsilon} + (T^{2+\varepsilon} K^{-3/4})^{11/14} (T^{3/2+\varepsilon} K^{11/4})^{3/14} \\ &\ll T^{53/28+\varepsilon}. \end{aligned}$$

For G_3, by a splitting argument and Lemma 5 again (notice $|\eta| \gg 1$) we get

$$G_3 \ll \frac{T^{3/2+\varepsilon}}{(NMKL)^{3/4}\delta} \sum_{\delta < |\eta| \le 2\delta, \delta \gg 1} 1 \tag{3.18}$$

$$\ll \frac{T^{3/2+\varepsilon}}{(NMKL)^{3/4}} K^{1/2} NML \ll T^{\frac{3}{2}+\varepsilon} y^{\frac{1}{2}} \ll T^{15/8+\varepsilon}.$$

Combining (3.6), (3.7), (3.10) and (3.15)-(3.18) we get

$$\int_T^{2T} S_2(x) dx \ll T^{53/28+\varepsilon}. \tag{3.19}$$

In the same way we can show that

$$\int_T^{2T} S_3(x) dx \ll T^{53/28+\varepsilon} \tag{3.20}$$

by Lemma 3 and Lemma 6.

From (3.2)-(3.5) , (3.19) and (3.20) we get

$$\int_T^{2T} \Delta^4(x) dx = \frac{3c_2}{32\pi^4} \int_T^{2T} x dx + O(T^{53/28+\varepsilon}), \tag{3.21}$$

which implies Theorem 1 immediately.

4 Preliminary Lemmas for Theorem 2

In order to prove Theorem 2 , we need the following Lemmas.

Lemma 7. We have

$$E(t) = \Sigma_1(t) + \Sigma_2(t) + O(\log^2 t)$$

with

$$\Sigma_1(t) := \frac{1}{\sqrt{2}} \sum_{n \le N} h(t,n) \cos(f(t,n)), \tag{4.1}$$

$$\Sigma_2(t) := -2 \sum_{n \le N'} d(n) n^{-1/2} (\log \frac{t}{2\pi n})^{-1} \cos(t \log \frac{t}{2\pi n} - t + \frac{\pi}{4}), \tag{4.2}$$

$$h(t,n) := (-1)^n d(n) n^{-1/2} (\frac{t}{2\pi n} + \frac{1}{4})^{-1/4} (g(t,n))^{-1}, \tag{4.3}$$

$$g(t,n) := arsinh((\frac{\pi n}{2t})^{1/2}), \tag{4.4}$$

$$f(t,n) := 2tg(t,n) + (2\pi n t + \pi^2 n^2)^{1/2} - \pi/4, \tag{4.5}$$

$$At \le N \le A't, N' := t/2\pi + N/2 - (N^2/4 + Nt/2\pi)^{1/2}, \tag{4.6}$$

where $0 < A < A'$ are any fixed constants.

Proof. This is the famous Atkinson's formula, see Atkinson [1] or Ivić[5, Theorem 15.1]. □

Lemma 8. Suppose $Y > 1$. Define

$$c_2^* : \quad = \sum_{\sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{l}} \frac{(-1)^{n+m+k+l} d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}},$$

$$c_2^*(Y) : \quad = \sum_{\substack{\sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{l} \\ n,m,k,l \le Y}} \frac{(-1)^{n+m+k+l} d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}},$$

$$c_2(Y) : \quad = \sum_{\substack{\sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{l} \\ n,m,k,l \le Y}} \frac{d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}}.$$

Then we have

$$c_2 = c_2^*, \quad c_2(Y) = c_2^*(Y), \quad |c_2 - c_2(Y)| \ll Y^{-1/2+\varepsilon}.$$

Proof. The estimate $|c_2 - c_2(Y)| \ll Y^{-1/2+\varepsilon}$ is a special case of Lemma 3.1 of the author[14] . The equalities $c_2 = c_2^*$ and $c_2(Y) = c_2^*(Y)$ follow from the fact that if $\sqrt{n_1} + \sqrt{n_2} = \sqrt{n_3} + \sqrt{n_4}$, then $n_1 + n_2 + n_3 + n_4$ must be an even number .

□

Lemma 9. Suppose $Y > 1$, then we have

$$H_1(Y) := \sum_{\substack{\sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{l} \\ n,m,k,l \le Y}} \frac{d(n)d(m)d(k)d(l)\max(n,m,k,l)^3}{(nmkl)^{3/4}} \ll Y^{5/2+\varepsilon}.$$

Proof. Suppose $\sqrt{n} + \sqrt{m} = \sqrt{k} + \sqrt{l}$, then we have

(1) $n = k, m = l$ or $n = l, m = k$;

or

(2)$n \ne k, l$.

If the case (2) holds, then by a classical result of Besicotitch, we know that

$$n = n_1^2 h, m = m_1^2 h, k = k_1^2 h, l = l_1^2 h, n_1 + m_1 = k_1 + l_1, \mu(h) \ne 0.$$

Thus we get

$$\begin{aligned}
H_1(Y) \quad & \ll \Sigma_1 + \Sigma_2, \\
\Sigma_1 \quad & \ll \sum_{n,k \le Y} \frac{d^2(n)d^2(m)\max(n,k)^3}{(nk)^{3/2}} \ll Y^{5/2}\log^3 Y, \\
\Sigma_2 \quad & \ll Y^\varepsilon \sum_{h<Y} \sum_{\substack{n_1+m_1=k_1+l_1 \\ n_1,m_1,k_1,l_1 \le Y^{1/2}h^{-1/2}}} \frac{\max(n_1,m_1,k_1,l_1)^6}{(n_1 m_1 k_1 l_1)^{3/2}} \\
& \ll Y^\varepsilon \sum_{h<Y} \sum_{\substack{n_1+m_1=k_1+l_1 \\ n_1,m_1,l_1 \le k_1 \le Y^{1/2}h^{-1/2}}} \frac{k_1^{9/2}}{(n_1 m_1 l_1)^{3/2}} \\
& \ll Y^\varepsilon \sum_{h<Y} \sum_{l_1} l_1^{-3/2} \sum_{\substack{n_1+m_1>k_1 \\ n_1,m_1 \le k_1 \le Y^{1/2}h^{-1/2}}} \frac{k_1^{9/2}}{(n_1 m_1)^{3/2}} \\
& \ll Y^\varepsilon \sum_{h<Y} \sum_{l_1} l_1^{-3/2} \sum_{n_1} n_1^{-3/2} \sum_{k_1 \ll m_1 \le k_1 \le Y^{1/2}h^{-1/2}} k_1^3 \\
& \ll Y^\varepsilon \sum_{h<Y} (Y^{1/2}h^{-1/2})^5 \ll Y^{5/2+\varepsilon}.
\end{aligned}$$

$\square$

Lemma 10. Suppose $Y > 1$, then we have

$$H_2(Y) := \sum_{\substack{\sqrt{n}+\sqrt{m}+\sqrt{k}=\sqrt{l} \\ n,m,k,l \le Y}} \frac{d(n)d(m)d(k)d(l)l^{3/4}}{(nmk)^{3/4}} \ll Y^{1/2+\varepsilon}.$$

Proof. If $\sqrt{n}+\sqrt{m}+\sqrt{k}=\sqrt{l}$, then we have

$$n = n_1^2 h, m = m_1^2 h, k = k_1^2 h, l = l_1^2 h, n_1 + m_1 + k_1 = l_1, \mu(h) \neq 0.$$

Thus we get

$$\begin{aligned} H_2(Y) \quad &\ll Y^\varepsilon \sum_{h(n_1+m_1+k_1)^2 \le Y} \frac{(n_1+m_1+k_1)^{3/2}}{h^{3/2}(n_1 m_1 k_1)^{3/2}} \\ &\ll Y^\varepsilon \sum_{h<Y} h^{-3/2} \sum_{n_1 \le m_1 \le k_1 \le Y^{1/2} h^{-1/2}} n_1^{-3/2} m_1^{-3/2} \ll Y^{1/2+\varepsilon}. \end{aligned}$$

$\square$

Lemma 11. Suppose $f_j(t)(1 \le j \le k)$ and $g(t)$ are continuous , monotonic real-valued functions on $[a,b]$ and let $g(t)$ have a continuous , monotonic derivative on $[a,b]$. If $|f_j(t)| \le A_j(1 \le j \le k), |g'(t)| \gg \Delta$ for any $t \in [a,b]$, then

$$\int_a^b f_1(t) \cdots f_k(t) e(g(t)) dt \ll A_1 \cdots A_k \Delta^{-1}.$$

Proof. This is Lemma 15.3 of Ivić[5]. $\square$

5 Proof of Theorem 2

Suppose $T \ge 10$. It suffices to evaluate $\int_T^{2T} E^4(t)dt$. Let $y := T^{1/3-\varepsilon}$. For any $T \le t \le 2T$, define

$$\mathcal{E}_1(t) := \frac{1}{\sqrt{2}} \sum_{n \le y} h(t,n) \cos(f(t,n)), \quad \mathcal{E}_2(t) := E(t) - \mathcal{E}_1(t).$$

From the inequality $(a+b)^4 - a^4 \ll |b|^3|a| + |b|^4$, we get

$$\int_T^{2T} E^4(t)dt = \int_T^{2T} \mathcal{E}_1^4(t)dt + O(\int_T^{2T} |\mathcal{E}_1(t)|^3 |\mathcal{E}_2(t)| dt) + O(\int_T^{2T} |\mathcal{E}_2(t)|^4 dt). \tag{5.1}$$

5.1 Evaluation of $\int_T^{2T} \mathcal{E}_1^4(t)dt$

In this subsection , we shall evaluate the integral $\int_T^{2T} \mathcal{E}_1^4(t)dt$. Similar to Tsang[11], we can write

$$\mathcal{E}_1^4(t) = \frac{3}{32}S_5(t) + \frac{3}{32}S_6(t) + \frac{1}{8}S_7(t) + \frac{1}{8}S_8(t) + \frac{1}{32}S_9(t), \tag{5.2}$$

where

$$\begin{aligned}
S_5(t) &:= \sum_{\substack{n,m,k,l\le y \\ \sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{l}}} H(t;n,m,k,l)\cos(F_1(t;n,m,k,l)),\\
S_6(t) &:= \sum_{\substack{n,m,k,l\le y \\ \sqrt{n}+\sqrt{m}\neq\sqrt{k}+\sqrt{l}}} H(t;n,m,k,l)\cos(F_1(t;n,m,k,l)),\\
S_7(t) &:= \sum_{\substack{n,m,k,l\le y \\ \sqrt{n}+\sqrt{m}+\sqrt{k}=\sqrt{l}}} H(t;n,m,k,l)\cos(F_2(t;n,m,k,l)),\\
S_8(t) &:= \sum_{\substack{n,m,k,l\le y \\ \sqrt{n}+\sqrt{m}+\sqrt{k}\neq\sqrt{l}}} H(t;n,m,k,l)\cos(F_2(t;n,m,k,l)),\\
S_9(t) &:= \sum_{n,m,k,l\le y} H(t;n,m,k,l)\cos(F_3(t;n,m,k,l)),\\
H(t;n,m,k,l) &:= h(t,n)h(t,m)h(t,k)h(t,l),\\
F_1(t;n,m,k,l) &:= f(t,n)+f(t,m)-f(t,k)-f(t,l),\\
F_2(t;n,m,k,l) &:= f(t,n)+f(t,m)+f(t,k)-f(t,l),\\
F_3(t;n,m,k,l) &:= f(t,n)+f(t,m)+f(t,k)+f(t,l).
\end{aligned}$$

We first estimate the integral $\int_T^{2T} S_5(t)dt$. For $n \le y$, it is easy to check that

$$h(t,n) = \frac{2^{3/4}}{\pi^{1/4}}\frac{(-1)^n d(n)}{n^{3/4}}t^{1/4}(1+O(\frac{n}{t})), \tag{5.3}$$

$$f(t,n) = 2^{3/2}(\pi nt)^{1/2} - \pi/4 + O(n^{3/2}t^{-1/2}), \tag{5.4}$$

$$f'(t,n) = 2^{1/2}(\pi n)^{1/2}t^{-1/2} + O(n^{3/2}t^{-3/2}). \tag{5.5}$$

If $\sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{l}$, then

$$\cos(F_1(n,m,k,l)) = \cos(O(\frac{D^{3/2}}{t^{1/2}})) = 1 + O(\frac{D^3}{t}), \tag{5.6}$$

where $D := \max(n, m, k, l)$. So from (5.3), (5.6), Lemma 8 and Lemma 9 we get

$$
\begin{aligned}
\int_T^{2T} S_5(t)dt &= \sum_{\substack{n,m,k,l\le y\\ \sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{l}}} \int_T^{2T} H(t;n,m,k,l)\cos(F_1(t;n,m,k,l))dt \\
&= \frac{8}{\pi} \sum_{\substack{n,m,k,l\le y\\ \sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{l}}} \frac{(-1)^{n+m+k+l}d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}} \\
&\quad \times \int_T^{2T} t(1+O(\frac{D}{t}))(1+(\frac{D^3}{t}))dt \\
&= \frac{8}{\pi} \sum_{\substack{n,m,k,l\le y\\ \sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{l}}} \frac{(-1)^{n+m+k+l}d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}} \\
&\quad \times \int_T^{2T} t(1+(\frac{D^3}{t}))dt \\
&= \frac{8}{\pi} \sum_{\substack{n,m,k,l\le y\\ \sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{l}}} \frac{(-1)^{n+m+k+l}d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}} \int_T^{2T} tdt \\
&\qquad + O(TH_1(y)) \\
&= \frac{8c_2}{\pi} \int_T^{2T} tdt + O(T^{1+\varepsilon}y^{5/2} + T^{2+\varepsilon}y^{-1/2}) \\
&= \frac{8c_2}{\pi} \int_T^{2T} tdt + O(T^{11/6+\varepsilon}).
\end{aligned}
\tag{5.7}
$$

Now we estimate $\int_T^{2T} S_6(t)dt$. From (5.5) we get

$$
F_1'(t;n,m,k,l) = (2\pi)^{1/2}\eta t^{-1/2} + O(D^{3/2}t^{-3/2}),
$$

where $\eta = \sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l}$. Write

$$
\int_T^{2T} S_6(t)dt = \int_{|\eta|\le T^{-1/2}} S_6(t)dt + \int_{|\eta|>T^{-1/2}} S_6(t)dt. \tag{5.8}
$$

If $|\eta| \le T^{-1/2}$, the by (5.3) and the trivial estimate we get

$$
\int_{|\eta|\le T^{-1/2}} S_6(t)dt \ll T^2 \sum_{\substack{n,m,k,l\le y;|\eta|\le T^{-1/2}\\ \sqrt{n}+\sqrt{m}\ne\sqrt{k}+\sqrt{l}}} \frac{d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}}. \tag{5.9}
$$

If $|\eta| > T^{-1/2}$, then $|F_1'(t;n,m,k,l)| \gg |\eta| T^{-1/2}$, thus from (5.3) and Lemma 11 we get

$$\int_{|\eta|>T^{-1/2}} S_6(t)dt \ll T^{3/2} \sum_{\substack{n,m,k,l\le y;|\eta|>T^{-1/2}\\ \sqrt{n}+\sqrt{m}\ne\sqrt{k}+\sqrt{l}}} \frac{d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}|\eta|}. \tag{5.10}$$

From (5.9), (5.10) and the estimate in Section 3 we get

$$\int_T^{2T} S_6(t)dt \ll \sum_{\substack{n,m,k,l\le y\\ \sqrt{n}+\sqrt{m}\ne\sqrt{k}+\sqrt{l}}} \frac{d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}} \min(T^2, T^{3/2}|\eta|^{-1}) \ll T^{53/28+\varepsilon}. \tag{5.11}$$

If $\sqrt{n}+\sqrt{m}+\sqrt{k}=\sqrt{l}$, then from (5.4) we have

$$F_2(t;n,m,k,l) = -\pi/2 + O(l^{3/2}t^{-1/2}), \ \ \cos F_2(t;n,m,k,l) \ll l^{3/2}t^{-1/2}.$$

Thus from (5.3), the trivial estimate and Lemma 10 we get

$$\int_T^{2T} S_7(t)dt \ll T^{3/2}H_2(y) \ll T^{3/2}y^{1/2+\varepsilon} \ll T^{5/3+\varepsilon}. \tag{5.12}$$

Similar to the integral $\int_T^{2T} S_6(t)dt$, we have

$$\int_T^{2T} S_8(t)dt \ll T^{53/28+\varepsilon}. \tag{5.13}$$

From (5.5) we get

$$F_3'(t;n,m,k,l) \gg (\sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{l})T^{-1/2},$$

which combining (5.3) and Lemma 11 implies

$$\int_T^{2T} S_9(t)dt \ll \sum_{n,m,k,l} \frac{d(n)d(m)d(k)d(l)T^{3/2}}{(nmkl)^{3/4}(\sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{l})} \ll T^{3/2+\varepsilon}y^{1/2} \ll T^{5/3+\varepsilon}. \tag{5.14}$$

From (5.2), (5.7), (5.11)-(5.14) we get

$$\int_T^{2T} \mathcal{E}_1^4(t)dt = \frac{3c_2}{4\pi}\int_T^{2T} tdt + O(T^{53/28+\varepsilon}). \tag{5.15}$$

5.2 Completion of proof of Theorem 2

Let $A_0 = 35/8$. Ivić[5, Thm 15.7] proved the estimate

$$\int_1^T |E(t)|^{A_0} dt \ll T^{1+A_0/4+\varepsilon}. \tag{5.16}$$

By his method we can show

$$\int_T^{2T} |\mathcal{E}_1(t)|^{A_0} dt \ll T^{1+A_0/4+\varepsilon}. \tag{5.17}$$

Thus

$$\int_T^{2T} |\mathcal{E}_2(t)|^{A_0} dt \ll T^{1+A_0/4+\varepsilon}. \tag{5.18}$$

We also have

$$\int_T^{2T} |\mathcal{E}_2(t)|^2 dt \ll T^{3/2+\varepsilon} y^{-1/2}, \tag{5.19}$$

which is the formula (4.15) of the author[14]. From (5.18) , (5.19) and the Hölder's inequality we get that the estimate

$$\int_T^{2T} |\mathcal{E}_2(t)|^A dt \ll T^{1+A/4+\varepsilon} y^{-(A_0-A)/2(A_0-2)} \tag{5.20}$$

holds for any $2 < A < A_0$. The details of the above estimates can be found in the author[14].

From (5.17), (5.20) and the Hölder's inequality we get

$$\begin{aligned} \int_T^{2T} |\mathcal{E}_1^3(t)\mathcal{E}_2(t)| dt &\ll \left(\int_1^T |\mathcal{E}_1(t)|^{A_0} dt\right)^{3/A_0} \left(\int_1^T |\mathcal{E}_2(t)|^{A_0/(A_0-3)} dt\right)^{(A_0-3)/A_0} \\ &\ll T^{2+\varepsilon} y^{-(A_0-4)/2(A_0-2)} \ll T^{2-19/108+\varepsilon}. \end{aligned} \tag{5.21}$$

From (5.1), (5.15), (5.20) with $A = 4$ and (5.21) we get

$$\int_T^{2T} E^4(t) dt = \frac{3c_2}{4\pi} \int_T^{2T} t dt + O(T^{53/28+\varepsilon}) \tag{5.22}$$

and Theorem 2 follows.

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